Chaos suppression in flows using proportional pulses in the system variables

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In this work a detailed analysis of a recently introduced chaos suppression method through proportional perturbations in the system variables [M.A. Matías and J. Güémez, Phys. Rev. Lett. **72**, 1455 (1994)] is presented. The method does not require any previous knowledge of the system dynamics and could be specially useful for those systems, e.g., chemical or biological, for which it may be advantageous to act on the system variables rather than on the parameters. The performance of the method is illustrated with several autonomous and nonautonomous flows, including issues such as the possibility of stabilizing different periodic or fixed point behaviors. Finally, a quantitative relationship among the parameters of the method is sought in terms of the highest Lyapunov exponent of the system. [S1063-651X(96)00507-7]

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I. INTRODUCTION

One century after the pioneering work of Poincaré, recent years have seen the emergence of the field of nonlinear dynamics (see, e.g., [1,2]) which consists in the study of far-from-equilibrium systems, characterized by responses that do not depend linearly on the applied stimulus. Some of the phenomena that are characteristic of this field, such as low-dimensional chaos, solitons, patterns, the emergence of complexity, etc., reflect a lot of unexpected order compared to the linear world of near-equilibrium systems, that are dominated by the tendency towards disorder dictated by the second law of thermodynamics. Nonlinear dynamics offers a common framework that is useful in a variety of different branches of science, ranging from fluid dynamics, meteorology, photonics, to biology, economics, and social sciences. One of the most fascinating behaviors of this kind of systems is low-dimensional deterministic chaos, in which a system with perfectly known evolution laws exhibits sensitive dependence on the initial conditions. Thus if one has two trajectories with initial conditions differing by some arbitrarily small amount, they will become completely uncorrelated after a certain time has passed. The consequence is a continuous loss of information on the dynamical behavior of the system.

After the practical discovery of chaos in a set of differential equations by Lorenz in 1963 [3], scientists have initially studied it as a curiosity, but, on the other hand, all efforts have been made to avoid the appearance of this behavior in practical settings. In fact, deterministic chaos may cause uncontrolled vibrations and fatigue failure in mechanical systems, temperature oscillations outside safe margins, voltage jumps in electrical systems, and, in general, malfunctioning and unpredictable behavior in these systems, including also chemical reactors. Despite its intrinsic complexity, the chaotic behavior exhibited by many experimental systems can be described in a low-dimensional way in terms of a few modes. The hallmark of those dissipative systems exhibiting chaos is the appearance of a fractal structure in state space, coming from a continuous stretch and fold process, called strange attractor, on which the asymptotic behavior of the system takes place.

The realization by Ott, Grebogi, and Yorke (OGY) [4] that one can apply small time-dependent perturbations to a chaotic system in such a way that its behavior becomes regular and predictable may look surprising at first sight. The key remark [5] is that a strange attractor can be viewed as the closure, or superposition, of a very large number of unstable periodic orbits. Intuitively one sees that in most routes to chaos new periodic motions are progressively created until the coexistence of all of them leads to chaos. This can be stated in other words by saying that the dynamical behavior of a chaotic system consists in a continuous switching between many different possible periodic behaviors, none of which predominates. The idea of controlling chaos is precisely to activate one of the underlying periodic behaviors (see [6–8] for recent surveys).

This simple idea is completely changing the reputation of chaotic systems, because under this perspective chaos is no longer a drawback, but rather an advantage. Within the context of classical linear control theory, see, e.g., [9], one needs to apply relatively large perturbations to the system in order to produce large changes in the system behavior. Although this philosophy has also been applied to nonlinear systems [10], the OGY scheme is more efficient, in the sense that the perturbations that one needs to apply to stabilize a given orbit are in the order of a few percent, due to the fact that the target periodic behavior is already present in the strange attractor. Moreover, one can choose among many different periodic behaviors and it is possible to switch from one to the other, implying this fact that controlled chaotic systems are very flexible. All this endeavor could be specially helpful to explain the behavior of many biological systems, for which the issues of adaptability to the environment and evolution

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may be explained if one sees them as self-regulatory nonlinear systems.

The OGY method [4,11,12] works in a discrete fashion, and when applied to N-dimensional flows consists in the definition of a suitably chosen $(N-1)$ Poincaré return map that gives the crossings of the flow with this plane. The method exploits the saddle character of the unstable periodic orbits and applies the necessary perturbation to a parameter of the system in such a form that the system always stays near the stable manifold of the periodic orbit. The perturbation to be applied to a system parameter is calculated by linearizing the flow locally around a trust ball. The method has proved its versatility in a number of exotic situations, such as the control of transient chaos [13] and the stabilization [14] of strange attractors destroyed in a crisis [15].

The OGY method has experienced a number of improvements that make it suitable for its application to experimental systems of which one only knows a time series [16], and to the study of higher-dimensional systems [17], among others. One variant of the chaos control idea, the occasional proportional feedback (OPF) method [18–20], exploits the strongly dissipative nature of the flows encountered in practical applications, allowing us to work with a one-dimensional return map of the system, which makes things much simpler. The result is that a very large number of applications to experimental settings exhibiting low-dimensional chaos and that use different approaches has appeared, ranging from a magnetoelastic ribbon [21], to spin-wave instabilities [22], a thermosiphon [23], a diode resonator [18], a laser [24], the Belousov-Zhabotinsky reaction [25], and also to heart tissue [26] and neurons [27]. These applications have enlarged the applicability of the controlling chaos ideas to a number of experimental situations (see also [28]).

One of the disadvantages of the OGY method in practical applications is that it is necessary to have a quite detailed knowledge of the system under study. For example, one needs to *learn* details about the location of the target unstable periodic orbit. A second source of potential problems is the discrete nature of the method: perturbations are only applied when the flow crosses the appropriate Poincaré plane. If the largest Lyapunov exponent of the system is relatively high the applied perturbations may be insufficient, and more refined techniques are required [29]. An approach that, in principle, overcomes the difficulty associated with the discrete nature of the OGY method is the continuous control technique put forward by Pyragas [30]. The idea in which this method is based is the synchronization of the target system with a time series produced by itself, either periodic or aperiodic. This method has also been applied to some experimental settings, including electrical circuits [31] and chaotic chemical reactions [32].

A different situation is that of the so-called nonfeedback methods, for which one uses no previous information about the system for controlling purposes. First of all, one has those methods [33–37] that apply an external modulated force on some system parameter, activating a regular behavior in the system in a resonant fashion. The main usefulness of these methods may be in the case of very fast systems, e.g., lasers, for which the natural time scale of the system is so fast that the application of a chaos control method like OGY, in which some calculations have to be carried out in order to determine the stable and unstable manifolds, cannot be performed in time. Another possibility is to apply an external high-frequency modulation [38] that stabilizes a periodic behavior in a nonresonant way, equivalent to the shift of some system parameter into a nonchaotic region.

All the methods presented so far, possibly with the exception of Pyragas' [30] method, have in common the fact that they are able to stabilize periodic behavior by acting on some system parameter. The aim of the present contribution is to discuss in some detail the properties of a recently introduced [39] chaos suppression method that works through the application of regular perturbations in the form of spikes, i.e., minute kicks, on the system variables. Like other nonfeedback methods, one of its advantages is the fact that it does not need a previous knowledge of the system's dynamical behavior. The fact that it acts on the system variables, rather than on some parameter, makes the present method specially attractive in the case of chemical or biological systems, for which finding a suitable parameter might be problematic. A direct implementation of the OGY method that acts on the system variables has also been introduced [40].

The method considered in the present work has already been applied to the case of dissipative one-dimensional [41] and two-dimensional (2D) maps (both invertible and noninvertible) [42]. The examples studied in Ref. [42] include the case of systems exhibiting a quasiperiodicity route to chaos [2], resulting from the coexistence of several periodic frequencies, and also the case of the systems in which a strange attractor is born or destroyed in a crisis [15]. This includes the case of the boundary crises, that happens when the strange attractor collides with a periodic orbit on its basin boundary, with the result that the size of the strange attractor changes suddenly (in the example studied in Ref. [42] points that lie outside the attractor's basin escape out to infinity, and this implies that the system goes to infinity after the crisis). The method has also been applied to the case in which the perturbations are applied to a Poincaré cross section of the flow [43].

In the present work the original method for flows [39], that typically applies several pulses between two crosses with a Poincaré plane, will be applied to different systems of differential equations, both autonomous and nonautonomous. In addition, quantitative relationships between the two parameters of the method, namely, the intensity of the perturbations and the time interval in which they are applied, will be presented. A crucial parameter in these relationships is the *degree of chaoticity* of the uncontrolled system, as measured by the highest Lyapunov exponent.

The present paper is organized as follows. Section II discusses the main features of the chaos suppression method used in the present work. Then, the method is applied in Sec. III to a few ordinary differential equation systems with the aim of illustrating the main features and potentialities of the method. Later, Sec. IV has the aim of exploring the method in a more quantitative fashion. Finally, Sec. V contains the main conclusions from the present work.

II. METHOD

The aim of the present work is to discuss the main properties of a recently introduced chaos suppression method

[39], namely, in the case that is applied to ordinary differential equation systems yielding deterministic chaos. The method works by applying instantaneous periodic kicks to the system variables, that amount to changes that are proportional to their current values, and that take the form,

$$X_i = X_i[1+\gamma_i \delta(t-j\tau)], \tag{1}$$

where X_i represents the ith variable of the system at a given instant of time, γ_i regulates the intensity of the perturbation applied to the ith variable, δ is Dirac's δ function, and j runs over natural numbers, implying that the kicks are applied at intervals that are uniformly spaced by τ. The proportional perturbations can be applied to all or only to some of the system variables. For example, in the case of nonlinear oscillators represented by second order differential equations, and where one has a pair of position and momentum coordinates, the perturbations might be applied to only one of them. Regarding the sign of the γ_i, it can be either positive or negative, as illustrated in more detail in the examples studied in the next section.

All the numerical integrations in this work have been performed by using a fourth-order fixed-step Runge-Kutta procedure [44]. The time step has been chosen to avoid spurious behavior, being typically in the range $\Delta t = 0.001$–0.01 units of time, unless otherwise stated. Fixing the value of the time step is quite convenient in the numerical work, because then the kicks are applied at fixed values of the number of integration steps, i.e., one makes $\tau = j\Delta t$, where j is some natural number. Otherwise, if one operates directly with time as a real variable spurious behavior may be obtained due to rounding errors.

The result of the operation of the method is that a different dynamical system is created that has γ and τ as parameters (although a relationship between them can be found, see Sec. IV). During $n-1$ steps the system evolves following the recipe of the unperturbed system, while at the end of the nth step a discrete change takes place. For a fixed value of τ, and depending on the value of γ, the dynamical system will exhibit chaos for low values of $|\gamma|$, until for a given *critical* value the behavior of the system will become regular, this transition being usually related to the routes toward chaos exhibited by the unperturbed dynamical system.

The stabilized periodic orbits (or fixed points) obtained by application of the method are not identical to the corresponding unstable periodic orbits (or fixed points) embedded in the strange attractor. Nevertheless, it has been empirically found, see, e.g., [42(b)], that these stabilized orbits (or fixed points) are close to orbits (or fixed points) of the unperturbed dynamical system for nearby parameter values that yield regular behavior. The method (1) exhibits some resemblance with those methods able to achieve chaos suppression through the application of an external resonant forcing term. However, it will be shown that if one fixes τ, then any periodicity can be stabilized by varying γ, implying that the method presents some analogies to the nonresonant chaos suppression method through fast modulation of some parameter of Ref. [38], that amounts to a shift in such a parameter and that allows stabilization of any periodic behavior.

III. APPLICATIONS

The present section contains a series of three-variable continuous models that exhibit deterministic chaos for some parameter ranges. The examples have been chosen to illustrate different features of the method, such as the possibility of using both negative and positive values of γ in (1), the stabilization of periodic orbits, but also of fixed points, and other features. In Sec. IV a more quantitative study of some features of the method will be presented.

A. The Rössler model

After studying the Lorenz attractor, Rössler was able to obtain the simplest nonlinear vector field capable of generating chaotic behavior [45] (see, however, [46]). This single-scroll strange attractor is written in the following form:

$$\dot{x} = -x-y, \quad \dot{y} = x+ay, \quad \dot{z} = b+z(x-c), \tag{2}$$

such that it has a single nonlinear term xz in $\dot{z}$.

By fixing a and b in the value $a=b=0.2$, one has a period-doubling (Feigenbaum) route to chaos where a period-2 orbit is created at $c=2.6$, and being $c\sim 4.2$ the accumulation point of the period doubling cascade, beyond which one has deterministic chaos, excepting for the presence of a number of periodic windows. The system has an unstable fixed point near the origin whose 2D unstable manifold presumably spans the strange attractor. It appears that the strange attractor does not exhibit a remerging tree (or period-doubling reversal) [47], at least for not too large values of c.

This system serves to illustrate the possibility of using either $\gamma<0$ or $\gamma>0$. At $c=4.6$ the system is chaotic, and chaos appears through a period-doubling route. Application of perturbations with $\gamma<0$ stabilizes different periodic behaviors, that correspond to regular states of the system for $c<4.6$. Thus, Fig. 1(a) presents a period-2 orbit obtained by using $\gamma=-0.004$, that is applied to all the variables, while period-1 or -4 orbits can be stabilized in the same way. Another possibility is to apply perturbations on only one or two of the variables. Purely empirical evidence shows that the most effective possibility in this case is to act only on y, and, thus, Fig. 1(b) presents a period-4 orbit stabilized by using $\gamma_y=-0.005$. It appears that there is no systematic procedure to determine *a priori* which variable is the most effective for this purpose, and analogous remarks for other models studied in the present work are based just on numerical evidence. It is possible to stabilize the system in a periodic behavior by using $\gamma>0$ values in the case that one is below a periodic window. Thus, for $c=5.0$ one is near a period-3 window and perturbations with $\gamma>0$ are able to stabilize this behavior, as shown in Fig. 1(c) for the case of $\gamma=0.002$. The robustness of the chaos suppression method has also been tested with similar models, namely, by considering two different chemical versions of Rössler's model [48,49].

B. The Hindmarsh-Rose model of a bursting neuron

The second example that we have chosen is that of the three-variable continuous model of a bursting neuron introduced by Hindmarsh and Rose in 1985 [50]. This study rep-

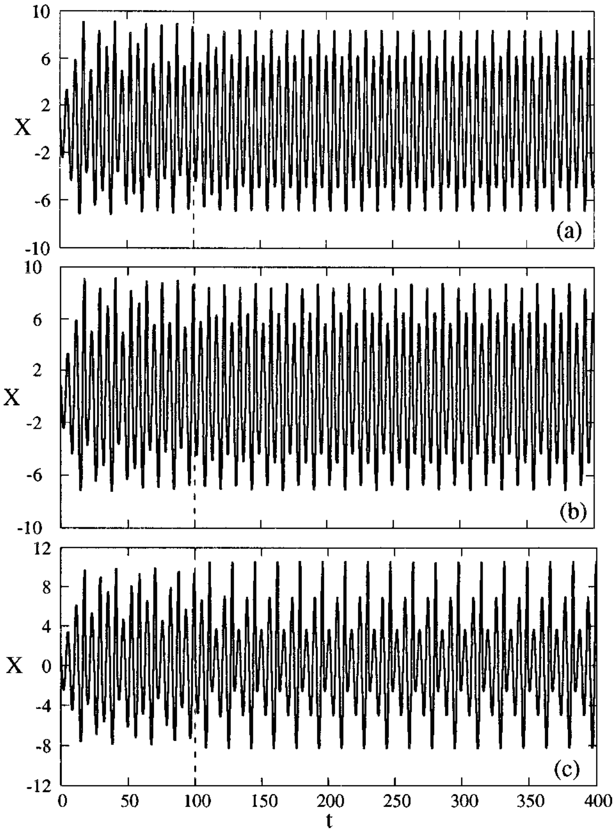

FIG. 1. Chaos suppression in Rössler's model, (2) and Ref. [45], by fixing the parameters $a=b=0.2$, being the time step for the integration $\Delta t=0.002$ and the interval between kicks in (1) $\tau=100\,\Delta t=0.2$ in all cases: (a) $c=4.6$ and $\gamma_x=\gamma_y=\gamma_z=-0.004$ (the system exhibits period-2 behavior); (b) $c=4.6$ and $\gamma_x=\gamma_z=0.0$, $\gamma_y=-0.005$ (the system exhibits period-4 behavior); (c) $c=5.0$, $\gamma_x=\gamma_y=\gamma_z=0.002$ (the system exhibits period-3 behavior).

resents a generalization of their previous work [51], based on the Fitzhugh's Bonhoeffer-van der Pol model [52]. The main idea is to have a model that produces action potentials separated by long interspike intervals, as found in real neurons, with the additional feature that the system may exhibit sustained bursting oscillations, i.e., a stable limit cycle instead of the usual quiescent (or fixed point) behavior. Furthermore, it can be shown that the model, that can be written in the form,

$$\dot{x}=y-ax^3+bx^2+I,\quad \dot{y}=c-dx^2-y,\quad \dot{z}=r[s(x-x_0)-z] \tag{3}$$

may exhibit deterministic chaos by adequately varying the I parameter [53].

A useful way of characterizing the behavior of the system is by using time-interval sequences for firing [53], studied as a function of I (that will be the bifurcation parameter). If one plots the time interval between spikes δ_n versus I, it can be seen that for increasing values of I, regular behavior with period-1, period-2 etc., is observed before chaos settles down. For still higher values of I one has periodic behavior again, this being an example of a period-doubling reversal or bubble [47]. We have chosen to work at $I=3.35$, inside the chaotic region. In analogy to the situation considered in Ref. [39] in the case of a model also exhibiting a period-doubling reversal behavior, one may obtain periodic behavior using either $\gamma<0$ or $\gamma>0$, where in the first case the observed behavior is related to the periodic behavior of the original system for $I>3.35$, while in the second case it is related to the behavior of the system for $I<3.35$.

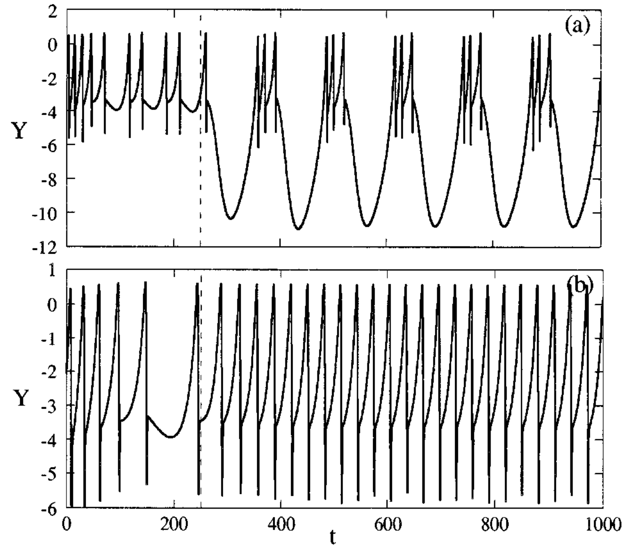

FIG. 2. Chaos suppression in the model of a bursting neuron due to Hindmarsh and Rose (3), and Ref. [50], where the parameters take the values $a=1.0$, $b=3.0$, $c=1.0$, $s=4.0$, $d=5.0$, $r=0.006$, $x_0=-1.6$, and $I=3.35$ [53], while the time step for integration is $\Delta t=0.02$ and in (1) $\tau=100\,\Delta t=2$: (a) $\gamma_x=\gamma_y=\gamma_z=0.004$ and one has a period-3 behavior; (b) $\gamma_x=\gamma_y=\gamma_z=-0.004$ and one has a period-1 behavior.

Thus Fig. 2(a) contains a stabilized period-3 orbit obtained with $\gamma=0.004$. Period-1, -2, and -4 orbits can be also stabilized with $\gamma>0$ values. In turn, Fig. 2(b) shows a stabilized period-2 orbit obtained with $\gamma=-0.000\,2$. Again, one can obtain the whole sequence with suitable values of $\gamma<0$. In brief, for systems presenting a remerging tree, like the three-variable autocatalator model [54] considered in Ref. [39] and the present model, one can stabilize periodic orbits of the system at both sides of the chaotic region in the bifurcation diagram by using either $\gamma<0$ or $\gamma>0$.

C. Proto-Lorenz and Lorenz models

In the original presentation [39] of the chaos suppression method (1), an application to the Lorenz model [3] was given. The result was that with $\gamma<0$ the method is able to stabilize one of the two fixed points from which the strange attractor emerges through the appearance of a homoclinic orbit, while which of the fixed points is obtained will depend on the initial conditions. In addition, the method is not able to stabilize purely periodic behavior in the *usual* chaotic region defined by the parameters $\sigma=10$, $R=28$, and $b=8/3$ [3]. The explanation for this result is that in this system there is not any nearby region in parameter space exhibiting periodic behavior, and, thus, the method (1) cannot stabilize this behavior.

Now we shall consider the so-called Proto-Lorenz model [55], obtained by transforming the original Lorenz flow, and

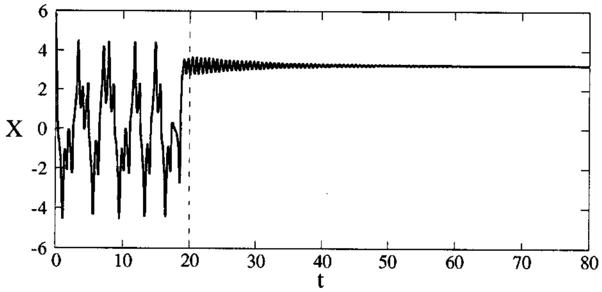

FIG. 3. Chaos suppression in the Proto-Lorenz system (4), and Ref. [55], for the parameter values $\sigma=10.0$, $b=8/3$, $r=28.0$, and with a time step for integration of $\Delta t=0.002$, while the parameters of (1) are $\gamma_x=\gamma_y=\gamma_z=-0.01$ and $\tau=20\,\Delta t=0.04$, being the behavior of the system of fixed point type.

where the strange attractor origins from a homoclinic orbit. The model can be written in the form,

$$\dot{x}=[-\sigma x^3+(2\sigma+r-z)x^2y+(\sigma-2)xy^2 -(r-z)y^3]/2(x^2+y^2),$$

$$\dot{y}=[(r-z)x^3+(\sigma-2)x^2y+(-2\sigma-r+z)xy^2 -\sigma y^3]/2(x^2+y^2),$$

$$\dot{z}=2x^3y-2xy^3-bz. \tag{4}$$

By working with the parameters $\sigma=10$, $b=8/3$, and $r=28$ this system exhibits chaotic behavior, and depending on the initial conditions any of the four unstable fixed points can be stabilized with $\gamma<0$ values, exactly in the same way as in the original Lorenz model. Thus in Fig. 3 one of the unstable fixed points of the system is stabilized with $\gamma=-0.01$. It is not possible to obtain this behavior by acting only on a subset of the system variables.

In this parameter region neither the original Lorenz model nor (4) exhibit a period-doubling route to chaos. Nevertheless, for the original Lorenz model,

$$\dot{x}=\sigma(y-x),\quad \dot{y}=Rx-y-xz,\quad \dot{z}=xy-bz, \tag{5}$$

it is possible to find regions of chaotic behavior near a period-doubling sequence and with periodic windows. In particular, we have fixed σ and b at the values $\sigma=1.43$ and $b=0.28$, while R may take values in the range $100<R<160$, these parameters appearing in the study of instabilities and chaos in laser dynamics [56]. In such a case, a strange attractor emerges from a fixed point at $R\sim 40$. By increasing R, periodic windows can be observed, for instance at $R\sim 110$, and at higher values of R a reverse period-doubling transition to chaos is observed. Thus, we have chosen to work at $R=130$, a region in which the system exhibits chaotic behavior sandwiched between periodic windows and a period-doubling sequence, with the aim of stabilizing this behavior through the use of the described method above (1).

In this case, the stabilization method works only if one applies perturbations on variable z. Thus Fig. 4(a) contains a stabilized period-1 orbit, that looks similar to the behavior obtained for $R\sim 160$. For the range of parameters considered by Lorenz, and also for the original model or the Proto-Lorenz model, it is possible to show that one can also stabilize the fixed point behavior by again acting only on z [see Fig. 4(b)]. The evidence for the fact that one gets the desired behavior by acting only on one of the variables is purely numerical, and, again, it is not possible to establish this fact *a priori*. Similar results can also be obtained for other systems that are related to the Lorenz model [48,49,57].

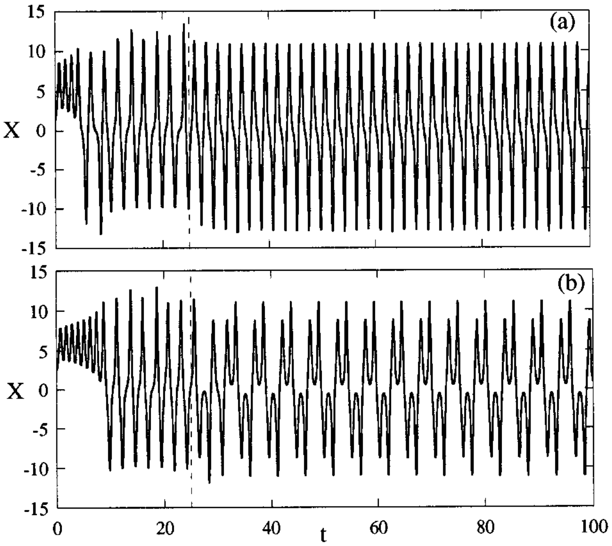

FIG. 4. Chaos suppression for the Lorenz model (5), and Ref. [3], with a set of parameters suitable for the study of laser instabilities: $\sigma=1.43$, $b=0.280$, and $R=130$, with a time step for integration of $\Delta t=0.002$, while in (1) $\tau=100\,\Delta t=0.2$: (a) $\gamma_x=\gamma_y=0.0$, $\gamma_z=-0.01$, being the final behavior a period-1 orbit; (b) $\gamma_x=\gamma_y=0.0; \gamma_z=0.01$, yielding a more complicated periodic orbit.

D. The Willamowski-Rössler model

The emergence of genuine low-dimensional chaos from a chemical mechanism, and not from hydrodynamical mixing effects, has long been under debate [58]. The use of simple models that are realistic from the chemical point of view, i.e., models in which the dynamical evolution equations can be written as a polynomial with at most quadratic terms, may shed light on this phenomenon. The Willamowski-Rössler [59] is probably the first model suggested in the literature that exhibits these features of realism from the chemical point of view. The model can be mathematically defined in the form,

$$\dot{x}=k_1x-k_{-1}x^2-k_2xy+k_2y^2-k_4xz+k_4,$$

$$\dot{y}=k_2xy-k_{-2}y^2-k_3y+k_{-3}, \tag{6}$$

$$\dot{z}=-k_4xz+k_{-4}+k_5z-k_{-5}z^2,$$

and it can be shown that this three-species chemical network exhibits complex oscillations for certain values of the parameters [60].

The dynamics of the Willamowski-Rössler model, with the parameters $k_1=30.0$, $k_{-1}=0.025$, $k_2=0.1$, $k_{-2}=0.000\,01$, $k_3=10.0$, $k_{-3}=0.01$, $k_{-4}=0.01$, $k_5=16.5$, $k_{-5}=0.05$ held fixed, and being k_4 the corresponding bifurcation parameter, can be described as follows

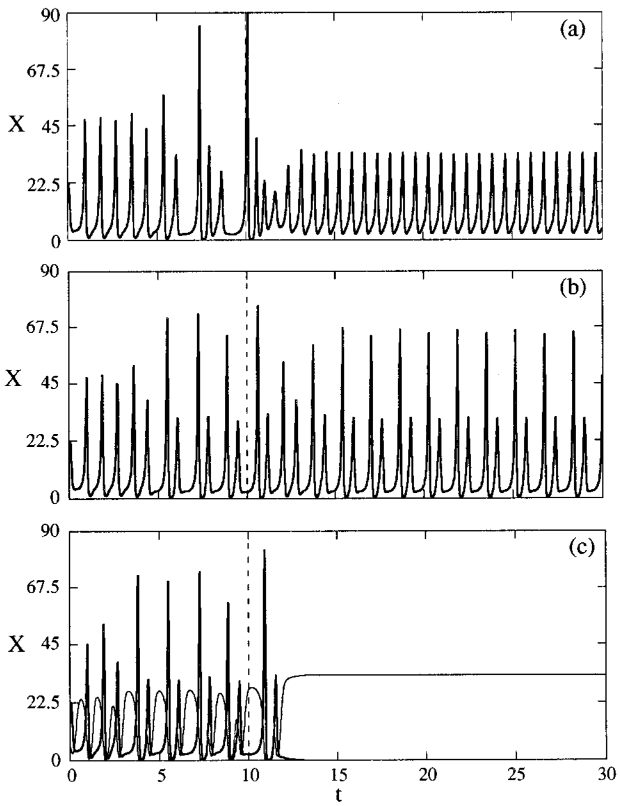

FIG. 5. Chaos suppression for the Willamowski-Rössler model (6), and Ref. [59], with the parameters $k_1=30.0$, $k_{-1}=0.25$, $k_2=1.0$, $k_{-2}=0.0001$, $k_3=10.0$, $k_{-3}=0.001$, $k_4=1.0$, $k_{-4}=0.001$, $k_5=16.5$, $k_{-5}=0.5$ [60] being the time step for the integration $\Delta t=0.0002$ and $\tau=200\,\Delta t=0.04$ in (1): (a) $\gamma_x=\gamma_y=\gamma_z=-0.02$ (that yields period-1 behavior); (b) $\gamma_y=\gamma_z=0.0$, $\gamma_x=0.04$ (that yields period-2 behavior); (c) $\gamma_x=\gamma_z=0.0$, $\gamma_y=0.04$ (that yields a transition to a stable fixed point coexisting with the strange attractor).

[60]. At $k_4=0.05$ one has a limit-cycle (period-1) behavior, that at $k_4=0.095$ becomes period 2, following a period-doubling route to chaos (the model exhibits chaos from $k_4\sim 0.097$). For still higher values of k_4 the system exhibits a period-doubling reversal [47], and one has a stable limit cycle for $k_4=0.13$, and finally a stable focus from $k_4=0.2$.

This system has been studied in the chaotic region, namely, by choosing $k_4=0.1$. By applying the above described chaos suppression method (1), it has been found that for $\gamma<0$ values it is possible to stabilize orbits related with the original system for $k_4<0.1$. Thus in Fig. 5(a) we present a stabilized period-1 orbit with $\gamma=-0.01$ and $\tau=0.04$, while period-2, period-4. and period-8 orbits are easily stabilized for other $\gamma<0$ values. It is also possible to stabilize periodic orbits by using $\gamma>0$ values, but not if the pulses are applied simultaneously to all the variables. Thus Fig. 5(b) presents a stabilized period-2 orbit obtained by using $\gamma_x=0.04$.

For the reported values of the parameters, the system presents a set of fixed points, of which two of them are stable, while the rest are unstable. The first stable fixed point has the coordinates $x_s=3.3338\times 10^{-4}$, $y_s=1.0000\times 10^{-4}$, $z_s=32.999$ [61], and by using adequate parameters in (1) it is possible to induce a transition in the system to this behavior, as can be seen from Fig. 5(b), where the perturbations have been applied only to y ($\gamma_y=0.04$). It has not been possible to stabilize any of the other fixed points, stable or unstable, reported in Ref. [61]. The stabilization of fixed point behavior occurs in this case through a transition from the strange attractor to the coexisting attractor, and bears no relationship to the stabilization of unstable fixed points reported for the Lorenz system in Sec. III C. In the case of the Willamowski-Rössler the role of (1) is analogous to the presence of noise in the system [61], while for the Lorenz system the method is effectively inducing a change in the parameters of the system, that leads it to a region in which the unstable fixed point becomes stable.

E. Duffing oscillator

Duffing's model is a very good example [62] of how deterministic chaos appears in many mechanical systems that may be described as oscillators deriving from a nonlinear potential. In this case the potential is quartic (has two wells), and damping and forcing terms are present. The system can be written in the form,

$$\ddot{x}+c\dot{x}+x^3-x=F\cos(\omega t). \tag{7}$$

It is also possible to write this model as a system of first-order differential equations by using the changes $\dot{x}=v$ and $z=\omega t$, in terms of the three state variables (x,v,z). By fixing the damping constant and driving frequency at the values $c=0.5$ and $\omega=1.0$, respectively, one obtains a route to chaos as a function of F. In the absence of external forcing $(F=0)$, the system has two stable fixed points at $x=\pm 1$, while when some forcing is applied the system oscillates with a frequency equal to the external frequency ω around these fixed points. In the region $0.309<F<0.321$ the system exhibits multistability, and beyond the upper limit a period-doubling route to chaos starts, that has its accumulation point at $F=0.3586$. For $F<0.386$ the system remains trapped at one of the wells, while at this value of F an attractor-merging crisis [15] occurs, and the strange attractor encompasses both wells.

The behavior observed for this system is very similar to that found for the case of the Holmes map, as reported in Ref. [42(b)] (indeed the Holmes map [63] was designed to be an approximation to the Poincaré map of the Duffing system). In this system one may apply (1) independently to any of the two variables x or $\dot{x}$, and Figs. 6(a) and (b) contain two examples in which periodic behavior is stabilized by acting on a single variable at a time.

F. Some remarks

Now we shall try to obtain some preliminary conclusions from the results obtained after the application of the chaos suppression method to the models studied in Sec. III. We hope that these caveats may be of help to potential users of the method. It is important to bear in mind that most of the conclusions mentioned in the present section emerge from empirical observations for the models studied in this work about the way in which the method (1) appears to work. It could be that one could find some counterexample in which some of these conclusions are violated. One of the observed

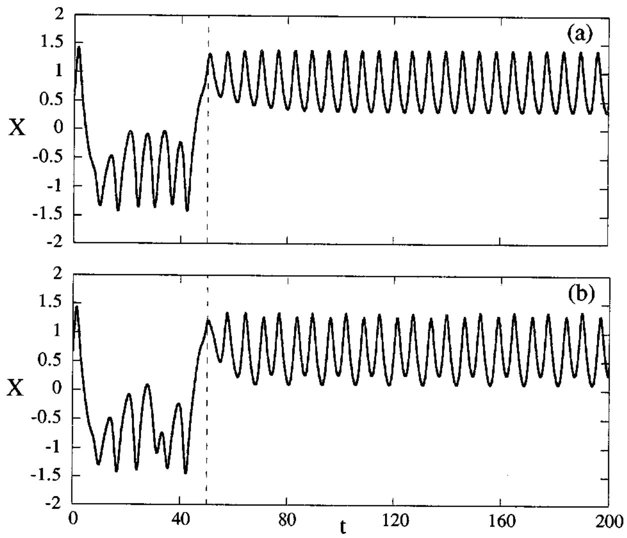

FIG. 6. Chaos suppression for the Duffing oscillator (7) with the parameters $c=0.5$, $F=0.42$, $\omega=1.0$ [62], being the time step for the integration $\Delta t=0.002$ and $\tau=100\,\Delta t=0.2$ in (1): (a) $\gamma_x=0$, $\gamma_v=-0.04$ and one gets period-1 behavior; (b) $\gamma_x=-0.03$, $\gamma_v=0$, that yields period-2 behavior.

features is that in the case $\gamma<0$ the method (1) works by stabilizing certain periodic orbits that coexist with the strange attractor. These unstable periodic orbits are similar, although not identical, to stable orbits corresponding to nearby regions of the system in parameter space for which the system exhibits a regular behavior. Instead, in the case that $\gamma>0$, the method apparently works in a safe way only when one has a bubble (or remerging tree). In such a case, the application of both $\gamma<0$ and $\gamma>0$ has the effect of stabilizing different regular behaviors related to one of the two periodic regions that are to the left and to the right of the bubble in the bifurcation diagram. The use of $\gamma>0$ in other cases typically yields transitions to an attractor at infinity (an explosion) or other nonuseful behaviors.

One could also wonder whether one should manipulate all the variables of the system, or rather a subset of them. By perturbing simultaneously all the variables of the system one typically obtains a regular behavior that is related to a behavior of the unperturbed system obtained by using different, but close, values of a system parameter. Instead, the cases in which perturbing a subset of the system variables one is able to stabilize different behaviors should be considered as exotic, in the sense that one obtains such behavior just for very specific systems (with the exception of the Lorenz model, as discussed in Sec. III C). Thus, in the case of the Willamowski-Rössler model (see Sec. III D), if one applies perturbations such that $\gamma_y<0$, while $\gamma_x=\gamma_z=0$, the effect is to induce a transition from the strange attractor to a coexisting fixed point. This is due to the particular structure of this model in phase space, as the strange attractor passes quite close to the origin in the x-y phase plane, while the coexisting stable fixed point has its y coordinate very close to zero, and, thus, the application of $\gamma_y<0$ has the effect of inducing this transition. However, one should not expect to obtain analogous results for other systems by applying this kind of specific perturbations to the system variables. Other examples in which the different variables may indeed have different physical interpretations, as is the case of the Duffing model of Sec. III E, offer another illustration of the application of perturbations to a subset of the system variables.

Another interesting feature of the present chaos suppression method is that, although formally the method works by introducing a new parameter γ in the system and then making fixed, static changes to that parameter, it effectively achieves some kind of dynamical exploration of the parameter space of the system. This can be better illustrated for the case of the Hindmarsh-Rose neuron model (see Sec. III B). The model is a function of eight parameters, and it is difficult to know *a priori* which is the most relevant parameter to be varied in order to obtain the whole bifurcation tree of the system. Indeed, by varying some of the parameters one gets a quite monotonic (and boring) behavior, completely unrelated to the sought transition between chaotic and periodic behaviors. In some sense, one needs to know a lot about the system, like which is the right parameter to be varied, before applying ordinary chaos control or chaos suppression methods on a system parameter. Instead, what we have observed for the systems considered in the present work is that if one uses (1) we have observed that, although γ is nothing else than a system parameter, one seems to be able to obtain the whole bifurcation tree of the system by varying it. This fact of *finding the right control parameter* is achieved without any previous knowledge about the system under study. These results should be considered as empirical findings for a number of systems, and it could be that other conclusions are obtained for other unexplored models.

It is in this context that the idea of *dynamical exploration* of the system should be understood, which may be a positive feature of the method when applied to situations for which a detailed modelization is not available, as is the usual situation when dealing with biological systems. At the present moment we cannot offer a theoretical explanation of why γ is so successful in achieving this exploration of the bifurcation tree, and we can only say that this is the behavior that is actually observed. The values of γ that have been used to suppress chaos are typically in the order of, at most, a few percent, as is the case of the present example. This illustrates the fact that the method only introduces small perturbations to the state of the system. In this sense, the method belongs to the *nonlinear chaos control* class of methods, and not to the class of methods based on a more linear way of reasoning, and that work by applying stronger perturbations to the state of the system.

IV. RELATIONSHIPS BETWEEN THE PARAMETERS OF THE CHAOS SUPPRESSION METHOD

In this section we shall try to analyze some generic features of the chaos suppression method through proportional changes in the system variables introduced in Sec. II, with the basic aim (that, however, is not always fulfilled) of obtaining some *a priori* information about the values of γ and τ to be used. In order to achieve this goal, it would be useful to have data for a number of systems, with the aim of obtaining some general trend. Thus we have performed a systematic study of the set of models recently introduced by Sprott [46]. This author has presented 19 models, supposed to be the simplest examples exhibiting chaotic behavior, with either five terms and two nonlinearities or six terms and one nonlinearity. In this work we shall not consider the first model given by Sprott (system A), because it is conservative, and we are only interested on dissipative systems.

In order to find quantitative relationships, it will be useful to consider the Lyapunov exponents, characterizing the growth of perturbations in the unstable direction of the attractor. Sprott has systematically reported the Lyapunov exponent for these systems [46]. Thus, if one makes a small perturbation $\delta x(0)$ on the attractor, its time evolution will be $\delta x(t) \sim \delta x(0)\exp(\lambda t)$, where λ is the highest Lyapunov exponent of the N-dimensional system under study, being this relationship only approximate. If one considers the situation in which the time scale in which the changes in the variables are applied, i.e., τ in (1), is faster than the time scale fixed by the inverse of the Lyapunov exponent, i.e., $\tau \ll t_L = 1/\lambda$, then one can linearize the previous expression to obtain,

$$\delta x(\tau) \approx \delta x(0)(1+\lambda\tau). \tag{8}$$

If one now linearizes $\delta x(t)$ for a time 2τ, and if $2\tau \ll t_L$ holds, the result is $\delta x(2\tau) \sim \delta x(0)(1+2\lambda\tau)$, and this implies that in this *linear* regime two successive applications of the method with γ and a time step τ are equivalent to a single application of (1) with time step 2τ and doubled intensity 2γ. This is equivalent to the previous empirical finding [39] that for a given system γ and τ are related one to each other in the form

$$\gamma/\tau = C, \tag{9}$$

C being a constant. This relationship has been shown to be fulfilled through numerical experimentation, provided that τ is not too large.

Another possibility that has been considered is that the proportional perturbations are applied to the system variables in the form of a square pulse, that has a finite duration, instead of doing it instantaneously. In the practical implementation of the method this would imply that the pulse elapses a finite number of integration steps. The results so far obtained for some of the examples considered in Sec. III indicate that the same stabilized orbits can be obtained by the appropriate tuning of the parameters. Moreover, a more quantitative relationship has been discovered between the duration of the square pulse, say $j\Delta t$, and the parameters γ and τ. If, in the spirit of (9), one fixes τ, then the relationship between the γ for the original method, where $j=1$, and its modified value with pulses of duration j time steps, say γ_j is found to be as simple as,

$$\gamma_j = \gamma/j. \tag{10}$$

This implies that through the application of the more realistic finite pulses one may reduce the relative impact of the perturbations making a smaller deformation on the system, although the total *amount* of the system variables that is injected or retired is constant.

One of the most rewarding aspects of deterministic chaos is the observed universality [2] in the way in which systems with completely different descriptions at the microscopic level perform the transition from regular to chaotic behavior. For the case of systems in the universality class of the logistic equation, i.e., of systems that exhibit a quadratic one-dimensional return map, and, thus, a period-doubling route to chaos, Huberman and Rudnick [64,65] have introduced the following relationship,

$$\lambda = \lambda_0 (p-p_c)^{\alpha}, \tag{11}$$

with $\alpha = \ln 2/\ln\delta = 0.4498$, and being $\delta = 4.52\ldots$ Feigenbaum's universal constant. In this relationship the highest Lyapunov exponent λ plays the role of an order (or, in this case, disorder) parameter of the system. Due to the existence of periodic windows in the midst of chaos, this relationship should be valid just for the envelope in which all the mergings from 2^n bands into 2^{n-1} bands occur inside the chaotic region.

Having a number of chaotic systems of the same type, we have chosen to look for some relationship between the Lyapunov exponent (disorder parameter) and just one characteristic parameter of the chaos suppression method (1) (as we have already shown that γ and τ are related). The chaos suppression method (1) works by stabilizing a given periodic orbit, but not in a smooth way due to the presence of instantaneous kicks to the variables. However, one can consider that the stabilized periodic orbit *shadows* a smooth periodic orbit of the system [39]. The orbits stabilized by the chaos suppression method (1) are a discrete approximation to continuous orbits. Taking into account relationship (9), one could replace a given realization of the chaos suppression method, that have given values for γ and $\tau = j\Delta t$, with another one in which the perturbations are applied at each integration step, and having $\gamma' = \gamma/j$. In this case the stabilized orbit would coincide with the shadow orbit. Of course, this discussion is only qualitative, as it is known that in some inherently unstable situations a shadow orbit cannot be defined [66]. A different situation is that in which the kicks are large, being τ probably also large, where (9) will not hold. In this case, the stabilized orbit would be clearly noncontinuous, and the relationship with the shadow orbit will be less clear. In such a case it is not even clear whether one can effectively stabilize a periodic orbit, even for structurally stable problems.

The idea now is that if one fixes the time step for the integration and the intensity of the perturbations, that will be the same for all the variables, then the interval between pulses should be related to the highest Lyapunov exponent of the system, that is the disorder parameter. This can be explained qualitatively as follows. If the highest Lyapunov exponent were zero, then no perturbations should be applied as the system would stay all the time in the periodic orbit, that would coincide with the *shadow* orbit. However, small positive Lyapunov exponents imply that one needs to *correct* the orbit from time to time in order to keep it close to the shadow orbit. In principle, the higher the Lyapunov exponent is, the smaller the interval between kicks should be.

We shall fix the intensity of the perturbation, that will be equal for all the variables, to take the value $\gamma = -0.02$ (2 % of perturbation), while the time step of the integration method takes the value $\Delta t = 0.01$, the one considered by Sprott in his study [46]. Then, the maximum possible value for the interval between perturbations for which a period-1 orbit is obtained will be recorded, with the aim of correlating

TABLE I. Largest Lyapunov exponent λ_+, characteristic time $\tau_+ = 1/\lambda_+$, and values of $\tau_1/\Delta t$, the maximum possible number of the interval between pulses, that are needed to stabilize period-1 orbits in selected Sprott chaotic models [46] (see [67] for the evolution equations) by using $\gamma_X = \gamma_Y = \gamma_Z = -0.02$ and $\Delta t = 0.01$ (see Sec. IV).

Case	λ_+	τ_+	$\tau_1/\Delta t$
D	0.103	9.7	17
F	0.117	8.5	14
G	0.034	29.4	26
H	0.117	8.5	15
I	0.012	83.3	138
J	0.076	13.1	46
K	0.038	26.3	46
L	0.061	16.4	24
M	0.044	22.7	36
N	0.076	13.1	50
O	0.049	20.4	35
P	0.087	11.5	26
Q	0.109	9.2	26
S	0.188	5.3	26

it with the the highest Lyapunov exponent (see Table I). The idea is to obtain a relationship that might allow us to predict the parameter to be applied in the chaos suppression method (1) for the unexplored systems from the knowledge of the highest Lyapunov exponent. Figure 7 shows three examples of the periodic orbits that are stabilized in these conditions for three of these systems, namely, systems D, I, and O.

The sought relationship between the highest Lyapunov exponent λ and the time interval between pulses to achieve a period-1 orbit $\tau_1/\Delta t$ has been obtained numerically by plotting the logarithm of the inverse of the Lyapunov exponent $\tau_+ = 1/\lambda$ versus the logarithm of $\tau_1/\Delta t$ (see Fig. 8 and Table I). Not all the systems studied in Ref. [46] have been considered in this study, as system A is conservative, systems B and C do not belong to the universality class of systems with a period-doubling route (these systems, that exhibit a double-scroll attractor, are in the same universality class as the Lorenz system). System E can only be stabilized by using $\gamma > 0$ values, and system R does not appear to yield a period-1 orbit (see Ref. [67] for the definition of the Sprott systems used in the present work). The result is a good fit to a straight line with a slope close to 1, that implies a straight line relationship between τ_+ and $\tau_1/\Delta t$, for eight of the systems, while the others depart somehow from this behavior (see Fig. 8).

One may wonder why the observed relationship is linear and does not have the form of (11). The fact that one finds a linear relationship between τ_+ and $\tau_1/\Delta t$, while τ_+ exhibits a relationship of the form (11) implies an analogous relationship for $\tau_1/\Delta t$, having exactly the same exponent α. This has been verified numerically for the systems considered in the present section. Another question is why some systems appear to fulfill so well this relationship, while the others significatively deviate from this behavior. One problem is that a single item of information is available for these systems, and, thus, neither the control parameter that regulates the route to chaos, nor the distance to the accumulation point

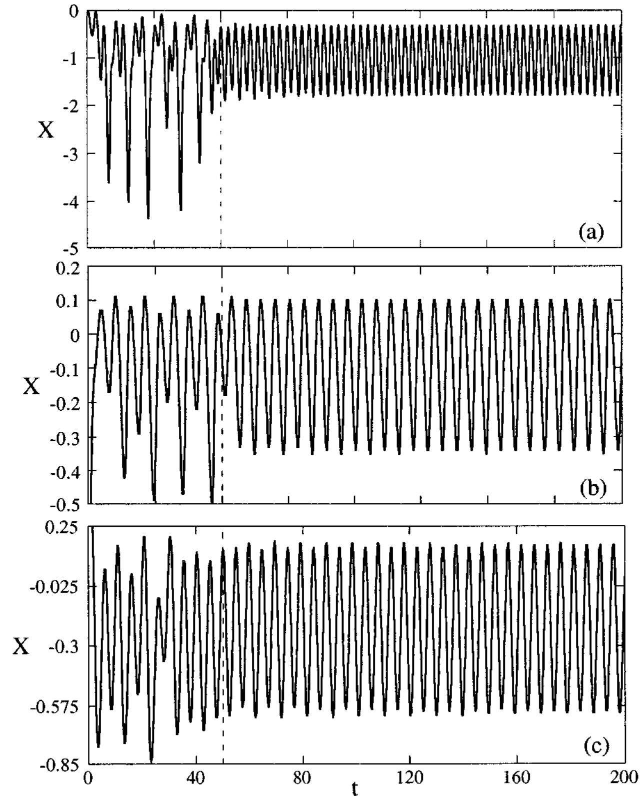

FIG. 7. Chaos suppression for some selected examples of the set of systems introduced by Sprott [46] (in all cases the parameters employed are those reported by Sprott, including the time step for the integration, that is $\Delta t = 0.01$). By fixing $\gamma_x = \gamma_y = \gamma_z = -0.02$ the maximum value of τ, called τ_1, such that one obtains period-1 behavior is reported (given in units of the time step): (a) model D with $\tau/\Delta t = 17$; (b) model I with $\tau/\Delta t = 138$; (c) model O with $\tau/\Delta t = 35$. Notice that these values are a part of Table I and of Fig. 8.

of Feigenbaum's route are known. As the Rössler system (see Sec. III A) is of the same type of the set of Sprott systems, we have decided to try to shed some light on these issues, as there exists a study of the variation of the highest Lyapunov exponent as a function of c [68]. If one considers several values of c in the range $4.2-5.0$, one indeed obtains a relationship of the form (11). By inserting these values as additional points in the previous fit, it can be seen that some of them are in agreement with the straight line fit, while some others fall somehow apart.

Moreover, this behavior is not erratic, as the parameter values that are close to the transition point do indeed yield a good fit, while some of the points that are farther from this transition point do not fit well. The key point is not just the distance from the transition, but rather whether the point under consideration is inside or near a periodic window in the chaotic regime. Huberman and Rudnick [64] found a relationship valid for the envelope past the transition point from periodic behavior to chaos. However, the chaotic region is interspersed with many periodic windows, and points in these regions will not obey the expected relationship between the Lyapunov exponent and their distance to the onset of chaos. Not much is known about the relative distance of the Sprott systems to the onset of chaos, and whether they

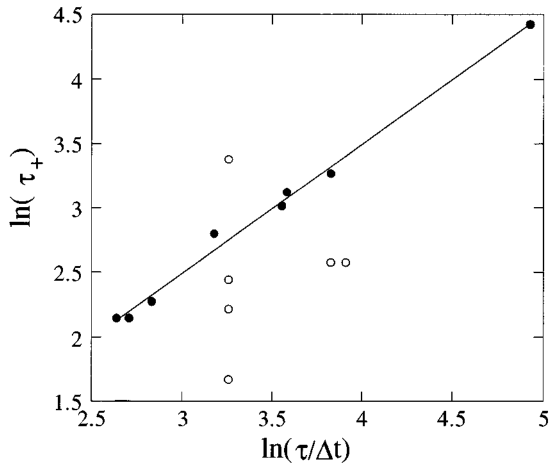

FIG. 8. Graphical representation of the values of Table I, to show the linear relationship $\ln\tau_+$ versus $\ln(\tau/\Delta t)$, being τ_+ the inverse of the highest Lyapunov exponent $\tau_+=1/\lambda$ and $\tau/\Delta t$ the maximum number of integration steps that are necessary to stabilize a period-1 orbit with $\gamma=-0.02$ and $\Delta t=0.01$. A straight line with parameters $\ln\tau_+=-0.521+1.004\ln\tau$ can be fitted very well to the eight systems represented with filled circles.

are close or not to a periodic window. Thus, finding quantitative relationships may be useful, but cannot be considered a universal panacea for chaotic systems in all possible parameter regimes.

To conclude this section, and taking Sprott models [46] as a useful example, it is perhaps interesting to mention one of the potential applications of the present method. When one studies for the first time a different chaotic model, knowing nothing about the behavior of the system when some parameter is varied, routes to chaos, etc., the chaos suppression algorithm suggested in this work may allow one to perform a useful first approach to the system. Thus, the method would afford some kind of exploration of the system through its *dynamical*, rather than by using the *static* parameters. In this sense, it would allow the exploration of the most typical behavior of the system (see also Sec. III F).

V. DISCUSSION AND CONCLUDING REMARKS

In the present work the recently suggested chaos suppression method [39] through proportional changes in the system variables (1), or, equivalently, minute kicks in the system variables that are proportional to their current value, has been applied to different kinds of systems and analyzed in some detail. Some of the conclusions obtained from this work, but also from other investigations from parallel studies, will now be summarized. First of all, what a potential user of the present method will probably wish to know is the answer to questions like the conditions under which the method will work or not, and which variables should be manipulated and in which form. The key point is that from the empirical evidence it appears that the method works by stabilizing periodic orbits or fixed points that are close, although not necessarily identical, to invariant sets for the unperturbed dynamical system, although for slightly different values of the parameters (see, e.g., [42(b)]). These orbits or fixed points become unstable when the system performs the transition to chaotic behavior. Thus, the main conclusion is that by applying proportional pulses simultaneously to all the system variables, with $\gamma<0$ and not too large values of γ and τ, one is able to stabilize some regular state of the system. This behavior is related to that exhibited by the original system in a nearby region in parameter space for which the system is regular. This should be the general recipe to be applied in a generic situation. Which behavior will turn out to be depends on the precise route to chaos of the system, as, in principle, the method seems not able to create behaviors that are not present in an unstable form in the system.

Other possible uses of the method should be regarded as more or less *exotic*, and they have been presented here only to illustrate other features of the method. Thus one may wonder which should be the sign of the perturbation γ to be applied in (1). One usually employs negative values for γ, that take the system to the prechaotic behavior, but in some circumstances, like the case of the systems exhibiting a remerging tree (or bubble) [47], the application of both positive and negative perturbations works in the aim of stabilizing the periodic behavior. Another point refers to the different performance obtained by applying perturbations to all the variables versus the application of pulses to a subset of these. It appears that, with the conspicuous exception of the Lorenz model (see Sec. III C), in almost all cases the application of changes to all the variables is more effective than the use of a subset of them.

In some cases, one only needs to apply stronger perturbations to a smaller number of variables, while in a few cases the method is no longer able to stabilize a periodic orbit. Another empirical remark is that the double-scroll strange attractors (like Lorenz model) are more difficult to stabilize than single-scroll strange attractors (like Rössler's model). A qualitative explanation for this fact might be as follows. Double-scroll strange attractors have a region between the two bands with a very strong sensitivity to initial conditions. Thus, one needs to perturb more the system in order that it becomes periodic, while the Lyapunov exponent will be higher, and all the arguments developed in Sec. IV should also apply. These are different manifestations of the fact that the single-scroll systems belong to the universality class of the logistic equation, while many double-scroll systems belong to that of the tent map (although not always).

The present proportional method appears to be superior to the alternative [43] in which one applies additive perturbations on the system variables. This remark should be understood in the following practical sense. The proportional method considered in the present work (1) is successful in suppressing chaos in a number of chaotic systems. Instead, the application of the additive perturbations is able to yield analogous results in many cases, while it fails for a number of examples. The opposite situation has not been found to hold in any case, i.e., it has not been found that the proportional method fails to suppress chaos for a system for which the additive method is able to achieve that goal. The explanation for the superior performance of the proportional method is probably that a chaotic system usually spans a substantial range of values in phase space, and the application of fixed, compromise, and additive perturbations on the system variables is less efficient than the use of proportional perturbations.

The method studied in the present work differs from the usual chaos control and chaos suppression methods [6,8] in that it does not focus on some accessible parameter (static) of the system, but instead has as its target the system variables (dynamical). For many physical systems locating a suitable system parameter is not difficult, and indeed it may be more amenable to manipulation than the system variables themselves (as is the case of lasers [56], where the system variables represent things like polarization, which is difficult to operate). Instead, if one considers chemical, biological, or other systems, locating the suitable parameters may be more difficult (in some cases one may even be completely ignorant about the evolution laws of the system). However, in these cases one can always identify at least some of the relevant variables of the system, with the meaning of populations, and alter them with the aim of controlling the behavior of the system and/or dynamically exploring other behaviors of the system. The meaning of *dynamical exploration* in this context would be as a variant of the usual *static exploration* carried out by pulling the knob on some parameter.

One example of the feasibility of this dynamical exploration is the Hindmarsh-Rose model considered in Sec. III B. This system (3) has eight different parameters, and, thus, a more or less systematic exploration of the behavior of the system would imply the consideration of variations in the eight-dimensional parameter space, which would be quite cumbersome. Instead, it is found that by applying proportional perturbations to the system variables through (1), and making equal perturbations by γ to all the variables, one may explore all the dynamical behaviors of the system with just a single parameter. The same can be said about the systems introduced by Sprott [46], that were used in Sec. IV. By choosing arbitrary values for the parameters and varying γ, one may have a first acquaintance with the behavior of the system, including the qualitative shape of the strange attractor, routes to chaos, alternative behaviors, etc. This idea of dynamical exploration can be useful regarding the role of the initial conditions, not just the system parameters. If one considers the Willamowski-Rössler model as an example (see Sec. III D), this system has two stable fixed points that coexist with the strange attractor. The point is that one of the fixed points appears to be globally more stable than the strange attractor, but its attraction basin is very small. This implies that it is very difficult that one becomes aware of the existence of the fixed point and its possible practical relevance if one works strictly at the deterministic level. However, if some noise is introduced into the system, there will be a noise-induced transition (see, e.g., [61]). As the chaos suppression method considered in this work is also able to induce this transition, one may consider it as a way of dynamically exploring phase space by applying kicks to the system variables, while one stays at the deterministic level of description.

Another field of interest in which it is foreseen that the present method might be of interest might be in the case of the extended systems exhibiting phenomena like that of spatiotemporal chaos, etc. If one has a discrete system, e.g., a coupled map lattice [69], the idea would be to apply perturbations to one of the nodes, that are diffusively coupled to the rest of the network, and then the regularization would extend to the rest of the network. A preliminary application of the present method along these lines has already been carried out in Ref. [70].

Chaotic systems have been usually considered as dangerous. While the use of chaos control techniques may be very helpful in keeping track of these situations, it appears that the common wisdom in medicine that deterministic chaos might be behind many diseases, like heart attacks, etc., could be unjustified. Recent evidence [71] indicates that chaos is healthy, because a living being lives in an adapting environment, and the repertoire of periodic behavior that is contained in a strange attractor makes the system more adaptable under external influences. Some theorists [72] even hold the view that maximum fitness is linked to a state that is in the boundary between order and chaos. Stabilized periodic behavior through a certain procedure might fulfill this requirement, and could be behind the way in which living beings, ecosystems, etc., achieve this productive state.

ACKNOWLEDGMENTS

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